# Inertia of Casimir energy

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Moving mirrors are submitted to reaction forces by vacuum fields. The motional force is known to vanish for a single mirror uniformly accelerating in vacuum. We show that inertial forces (proportional to accelerations) arise in the presence of a second scatterer, exhibiting properties expected for a relative inertia: the mass corrections depend upon the distance between the mirrors, and each mirror experiences a force proportional to the acceleration of the other one. When the two mirrors move with the same acceleration, the mass correction obtained for the cavity represents the contribution to inertia of Casimir energy. Accounting for the fact that the cavity moves as a stressed rigid body, it turns out that this contribution fits Einstein's law of inertia of energy.

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Scatterers in vacuum are submitted to the radiation pressure of vacuum fields. In a configuration with two motionless mirrors, a mean force, the so-called Casimir force, results for each of them [1,2]. As known since Einstein [3], the field energy stocked inside a box contributes to its inertia. This law of inertia of energy has to be valid for any kind of energy bound to the motion of any system [4]. Casimir energy corresponds to the particular case of a Fabry-Perot cavity, a system formed by two mirrors which models Einstein's "box", immersed in vacuum fields, and can be considered as a small amount of vacuum energy stocked inside the cavity, actually a negative amount as a binding energy for an atomic system. As a consequence of the law of inertia of energy, the inertial mass of the cavity has to vary with Casimir energy for example, when the length is varied.

Because of the infiniteness of the vacuum energy density (when integrated over frequency) however, it has often been claimed that vacuum energy is not a real energy like photons. Particularly, it seems to be admitted that it does not gravitate [5] so that its contribution to inertial forces can also be questionned. Nevertheless, it has also been argued that Casimir energy, a finite energy difference between two vacuum configurations, has to contribute to gravitation and inertia [6]. In the present paper, we demonstrate that Casimir energy does indeed contribute to inertia of the Fabry-Perot cavity. For this demonstration, we use previously obtained results providing the motional forces, i.e. the mean forces experienced by mirrors moving in vacuum [7,8] which are associated with the quantum fluctuations of vacuum radiation pressure [9].

For a perfectly reflecting mirror alone in the vacuum state of a scalar field in a two dimensional (2D) spacetime [10], the motional force $\delta F$ can be written in a linear approximation in the displacement $\delta q$ as

$$\delta F(t) = \frac{\hbar}{6\pi c^2}\delta q'''(t) \tag{1}$$

This force vanishes for a uniform velocity, as well as for a uniform acceleration, which can be interpreted as a consequence of spatial symmetries of the vacuum. Vacuum fields are invariant under the action of Lorentz boosts [11], so that the motional force vanishes for uniform velocity. When seen by a uniformly accelerating observer, vacuum fields appear as thermal fields for a motionless observer [12] and the motional force vanishes for uniform acceleration. These properties remain true for a partially transmitting mirror in vacuum [7,13].

The motional forces have also been computed in the configuration of a Fabry-Perot cavity in vacuum [10,8], where the spatial symmetries previously discussed are broken. We show in the present paper that they contain inertial forces, proportional to the accelerations, which exhibit properties expected for a relative inertia: the mass corrections depend upon the distance, and forces are obtained for each mirror, which are proportional to the acceleration of the other one. We will also compute the mass correction for a global motion of the system. In the limiting case of perfect mirrors for instance, we obtain a mass correction which is twice the value of Casimir energy over $c^2$. Taking into consideration that the cavity moves as a stressed rigid body (same motion for the two mirrors), a situation elucidated by Einstein himself [14], it turns out that this is exactly the prediction of the law of inertia of energy.

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## PERFECT MIRRORS: QUALITATIVE DERIVATION OF INERTIA CORRECTIONS

We first study the situation of two perfect point-like mirrors in the vacuum state of a 2D scalar field. All relations, including the definition of vacuum, will refer to an inertial frame.

A linear approximation of the expression obtained in this case by Fulling and Davies [10] provides us with the force $\delta F_1$ exerted upon the mirror 1 as a function of the positions $\delta q_1$ and $\delta q_2$ of the two mirrors [15]; more precisely, $\delta F_1$ is the response of the mean force to classical displacements)

$$\begin{aligned}\delta F_1(t) = &\frac{\hbar}{6\pi c^2}\left(\delta q_1'''(t) - \delta q_2'''(t-\tau) + \delta q_1'''(t-2\tau) - \delta q_2'''(t-3\tau) + \ldots\right) \\ &+\frac{\hbar\pi}{6c^2\tau^2}\left(\frac{1}{2}\delta q_1'(t) - \delta q_2'(t-\tau) + \delta q_1'(t-2\tau) - \delta q_2'(t-3\tau) + \ldots\right)\end{aligned} \tag{2}$$

where $\tau$ is the propagation delay of light from one mirror to the other and $q$ the distance between the mirrors

$$\tau = \frac{q}{c}$$

The terms proportional to third order derivatives in equation (2) have the same form as the damping force (1) for a single mirror, but the modification of the stress tensor generated by the mirrors' motion now propagates from one mirror to the other and is reflected back by the mirrors. This is why the time of flight $\tau$ appears in the expression of the force. Although they have the same form as for a single mirror, these terms give rise to mass corrections. A qualitative demonstration goes as follows. Extracting the contribution of these terms to the motional force (first line of eq. 2) and considering the quasistatic limit where the positions vary slowly on a time scale $\tau$, one transforms the sum over discrete times into an integral

$$\delta F_1(t) \approx \int^t \mathrm{d}t' \frac{\hbar}{12\pi c^2\tau}\left(\delta q_1'''(t') - \delta q_2'''(t')\right)$$

and one gets a motional force depending upon the accelerations of the two mirrors

$$\delta F_1(t) \approx \frac{\hbar}{12\pi cq}\left(\delta q_1''(t) - \delta q_2''(t)\right)$$

The other terms of equation (2), proportional to velocities, are not present in the one mirror problem. They are associated with the existence of a static Casimir force, since their contribution (second line of eq. 2) leads in the quasistatic approximation to

$$\delta F_1(t) \approx \frac{\hbar c\pi}{12q^3}\left(\delta q_1(t) - \delta q_2(t)\right)$$

This is the variation with the distance $q = q_2 - q_1$ of the mean Casimir force $F_1$

$$F_1 = \frac{\hbar c\pi}{24q^2} = \partial_q U \qquad U = -\frac{\hbar c\pi}{24q} \tag{3}$$

where $U$ is the known expression for the Casimir energy [2].

In the approximate expressions obtained in this section, the position or acceleration of one mirror are measured relatively to the position or acceleration of the other one. In the more precise discussion which follows, this property will remain true for the positions (the static force only depends upon the distance between the two mirrors), but accelerations will appear only partly as relative quantities; this feature will lead to a non vanishing correction for the global mass of the cavity.

## PERFECT MIRRORS: QUANTITATIVE EVALUATION OF INERTIA CORRECTIONS

The motional forces $\delta F_i$ can be written in the temporal or spectral domains

$$\delta F_i(t) = \int \mathrm{d}\tau \sum_j \chi_{ij}(\tau)\delta q_j(t-\tau)$$

$$\delta F_i[\omega] = \sum_j \chi_{ij}[\omega]\delta q_j[\omega]$$

where we denote for any function $f$

$$f(t) = \int \frac{\mathrm{d}\omega}{2\pi} f[\omega] e^{-i\omega t}$$

The expressions (2) for the motional forces correspond to the following susceptibility functions where the real and imaginary parts are separated in the spectral domain

$$\chi_{ij}[\omega] = \chi_{ji}[\omega] = \widetilde{\xi}_{ij}[\omega] + i\xi_{ij}[\omega] \tag{4a}$$

$$\xi_{11}[\omega] = \xi_{22}[\omega] = \frac{\hbar}{12\pi c^2}\omega^3 \tag{4b}$$

$$\xi_{12}[\omega] = 0 \tag{4c}$$

$$\widetilde{\xi}_{11}[\omega] = \widetilde{\xi}_{22}[\omega] = -\frac{\hbar}{12\pi c^2}\frac{\omega^3 - \omega\frac{\pi^2}{\tau^2}}{\tan(\omega\tau)} \tag{4d}$$

$$\widetilde{\xi}_{12}[\omega] = \frac{\hbar}{12\pi c^2}\frac{\omega^3 - \omega\frac{\pi^2}{\tau^2}}{\sin(\omega\tau)} \tag{4e}$$

It can be noted that the dissipative parts $\xi_{ij}$, imaginary parts of $\chi_{ij}$ and odd functions of $\omega$, coincide with the contributions of the outer space (for each mirror, only one half of the outer space contributes and $\xi_{ii}$ is half the value of $\xi$ for a single mirror) while the dispersive parts $\widetilde{\xi}_{ij}$, real parts of $\chi_{ij}$ and even functions of $\omega$, are the contributions of the intracavity space. This fact has a clear interpretation: the outer fields constitute an open quantum system, corresponding to a continuous spectrum; in contrast, the intracavity fields are characterized by a discrete spectrum and are unable to contribute to dissipation. As a consequence, there is no dissipative part in the mutual susceptibility ($\xi_{12} = 0$).

The dissipative parts $\xi_{ij}$ are the commutators of the force operators and can be deduced from the correlation function $C_{ij}$ computed for motionless mirrors [7,8]

$$\xi_{ij}(t) = \frac{\langle [F_i(t), F_j(0)] \rangle}{2\hbar} = \frac{C_{ij}(t) - C_{ji}(-t)}{2\hbar} \qquad C_{ij}(t) = \langle F_i(t) F_j(0) \rangle - \langle F_i \rangle \langle F_j \rangle$$

Fluctuations can also be recovered from dissipation through the relation [7,8]

$$C_{ij}[\omega] = 2\hbar\theta(\omega)\xi_{ij}[\omega]$$

i.e. the fluctuation-dissipation relation [16] at the limit of zero temperature.

The dispersive functions $\widetilde{\xi}_{ij}$ diverge at the zeros $\omega = m\frac{\pi}{\tau}$ of the denominators, except for $m = 0$ or $m = 1$ where the numerators vanish. These divergences result from a constructive interference between the different numbers of cavity roundtrips [8]. According to causality (which is apparent in eqs 2), the susceptibility functions (4) are analytic in the upper half-plane of the frequency domain ($\mathrm{Im}\,\omega > 0$), and the dispersive parts are related to the dissipative ones through dispersion relations, a property which is more easily checked for partially transmitting mirrors (see the next section).

As the susceptibility functions are regular around $\omega = 0$, a quasistatic expansion of the force may be performed, in which coefficients are introduced for describing static, viscous and inertial forces (we will not be interested in higher order quasistatic coefficients)

$$\delta F_i(t) = -\sum_j \left( \kappa_{ij}\delta q_j(t) + \lambda_{ij}\delta q_j'(t) + \mu_{ij}\delta q_j''(t) + \ldots \right) \tag{5a}$$

$$\chi_{ij}[\omega] = -\kappa_{ij} + i\omega\lambda_{ij} + \omega^2\mu_{ij} + \ldots \tag{5b}$$

$$\kappa_{ij} = -\chi_{ij}[0] \qquad \lambda_{ij} = -i\chi_{ij}'[0] \qquad \mu_{ij} = \frac{\chi_{ij}''[0]}{2} \tag{5c}$$

The quasistatic coefficients are deduced from equations (4); the static coefficients $\kappa_{ij}$ describe the variation with the distance $q$ of the force $F = F_1 = -F_2$

$$\kappa_{11} = \kappa_{22} = -\kappa_{12} = -\kappa_{21} = -\frac{\hbar c\pi}{12q^3} = \frac{\mathrm{d}F}{\mathrm{d}q} \tag{6a}$$

The viscosity coefficients $\lambda_{ij}$ vanish while the inertia corrections $\mu_{ij}$ are given by

$$\mu_{11} = \mu_{22} = -\frac{\hbar}{12\pi c q}\left(1+\frac{\pi^2}{3}\right) \tag{6b}$$

$$\mu_{12} = \mu_{21} = -\frac{\hbar}{12\pi c q}\left(-1+\frac{\pi^2}{6}\right) \tag{6c}$$

The mass corrections differ from the approximate ones previously discussed by numerical factors: the terms in equations (2) which are proportional to velocities also contribute to the inertial corrections; $\mu_{11}$ and $\mu_{12}$ no longer have opposite values and they actually have the same sign. This will allow the mass correction for a global motion of the cavity to differ from zero.

## PARTIALLY TRANSMITTING MIRRORS

In a more satisfactory treatment, perfect mirrors are replaced by partially transmitting mirrors, described by reflection and transmission amplitudes respectively denoted $r_i$ and $s_i$ for the mirror $i = 1, 2$ obeying unitarity, causality and high frequency transparency requirements [2]. These mirrors are more easily shown to obey causality [7], the divergences associated with the infiniteness of vacuum energy are regularized [2,8], and the stability problem arising for a perfect mirror may be solved [17]. A resonant enhancement of the motional Casimir force subsists at the optical resonance frequencies of the Fabry-Perot cavity [8,18].

The susceptibility functions $\chi_{ij}$ may be written (see eqs 20 and 21 of ref. [8]; $\varepsilon$ is the sign function)

$$\chi_{ij}[\omega] = \frac{i\hbar}{2c^2}\int \frac{\mathrm{d}\omega'}{2\pi}\omega'(\omega-\omega')\varepsilon(\omega')\gamma^R_{ij}[\omega', \omega-\omega']$$

where the coefficients $\gamma^R_{ij}$ are the sum of two parts

$$\gamma^R_{ij}[\omega,\omega'] = \gamma^S_{ij}[\omega,\omega'] + \gamma^A_{ij}[-\omega,\omega']$$

Both functions $\gamma^S_{ij}$ and $\gamma^A_{ij}$ are symmetrical in the exchange of their two parameters, so that one obtains

$$\begin{aligned}
\chi_{ij}[\omega] &= \chi^S_{ij}[\omega] + \chi^A_{ij}[\omega] \\
\chi^S_{ij}[\omega] &= \frac{i\hbar}{2c^2}\int_0^\omega \frac{\mathrm{d}\omega'}{2\pi}\omega'(\omega-\omega')\gamma^S_{ij}[\omega',\omega-\omega'] \\
\chi^A_{ij}[\omega] &= \frac{i\hbar}{2c^2}\int_0^\infty \frac{\mathrm{d}\omega'}{2\pi}\omega'\left((\omega+\omega')\gamma^A_{ij}[\omega',\omega+\omega'] + (\omega-\omega')\gamma^A_{ij}[-\omega',\omega-\omega']\right)
\end{aligned}$$

The functions $\chi^S_{ij}$ scale as $\omega^3$ in the vicinity of zero frequency as the susceptibility function for a single mirror, and they do not contribute to the quasistatic coefficients $\kappa_{ij}$, $\lambda_{ij}$ and $\mu_{ij}$, which we are interested in. We will not discuss them in more detail. The functions $\gamma^A_{ij}$ are given by

$$\begin{aligned}
\gamma^A_{11}[\omega,\omega'] &= \frac{(r_1[\omega]+r_1[\omega'])\left(r_2[\omega]e^{2i\omega\tau} + r_2[\omega']e^{2i\omega'\tau}\right)}{d[\omega]d[\omega']} \\
\gamma^A_{21}[\omega,\omega'] &= -\frac{(r_1[\omega]+r_1[\omega'])\,(r_2[\omega]+r_2[\omega'])\,e^{i(\omega+\omega')\tau}}{d[\omega]d[\omega']} \\
d[\omega] &= 1 - r_1[\omega]r_2[\omega]e^{2i\omega\tau}
\end{aligned}$$

$\gamma^A_{22}$ and $\gamma^A_{12} = \gamma^A_{21}$ are obtained by exchanging the roles of the two mirrors. Straightforward differentiations of the functions $\chi^A_{ij}$ then lead to the quasistatic coefficients (see eqs 5)

$$\begin{aligned}
\kappa_{ij} &= -\frac{i\hbar}{2c^2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega^2\left(\gamma^A_{ij}[\omega,\omega] - \gamma^A_{ij}[-\omega,-\omega]\right) \\
\lambda_{ij} &= \frac{\hbar}{2c^2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega\left(\gamma^A_{ij}[\omega,\omega] + \gamma^A_{ij}[-\omega,-\omega] + \omega\gamma^{A\prime}_{ij}[\omega,\omega] - \omega\gamma^{A\prime}_{ij}[-\omega,-\omega]\right) \\
\mu_{ij} &= \frac{i\hbar}{4c^2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega\left(2\gamma^{A\prime}_{ij}[\omega,\omega] + 2\gamma^{A\prime}_{ij}[-\omega,-\omega] + \omega\gamma^{A\prime\prime}_{ij}[\omega,\omega] - \omega\gamma^{A\prime\prime}_{ij}[-\omega,-\omega]\right)
\end{aligned}$$

It is understood that differentiation only bears on one of the two frequency parameters

$$\gamma_{ij}^{A\prime}[\omega,\omega] = \left(\partial_\omega \gamma_{ij}^{A}[\omega,\omega']\right)_{\omega'=\omega} = \frac{1}{2}\frac{\mathrm{d}\gamma_{ij}^{A}[\omega,\omega]}{\mathrm{d}\omega}$$
$$\gamma_{ij}^{A\prime\prime}[\omega,\omega] = \left(\partial_\omega^2 \gamma_{ij}^{A}[\omega,\omega']\right)_{\omega'=\omega}$$

One checks [8] that the static coefficients $\kappa_{ij}$ fit the variation of the mean Casimir force $F$ between the two partially transmitting mirrors

$$\kappa_{11} = \kappa_{22} = -\kappa_{12} = -\kappa_{21} = \frac{\mathrm{d}F}{\mathrm{d}q} \tag{7a}$$
$$F = \frac{\hbar}{c}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega\left(1 - \frac{1}{d[\omega]} + 1 - \frac{1}{d[-\omega]}\right) \tag{7b}$$

The viscosity coefficients $\lambda_{ij}$ remain equal to zero for partially transmitting mirrors

$$\lambda_{ij} = 0 \tag{8}$$

One eventually rewrites the inertia corrections

$$\mu_{ij} = \frac{i\hbar}{4c^2}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega^2\left(\Gamma_{ij}[\omega] - \Gamma_{ij}[-\omega]\right) \tag{9a}$$
$$\Gamma_{ij}[\omega] = \gamma_{ij}^{A\prime\prime}[\omega,\omega] - \frac{\mathrm{d}\gamma_{ij}^{A\prime}[\omega,\omega]}{\mathrm{d}\omega} = -\left(\partial_\omega\partial_{\omega'}\gamma_{ij}^{A}[\omega,\omega']\right)_{\omega'=\omega}$$
$$\Gamma_{11}[\omega] = -2\frac{r_1'[\omega]e^{2i\omega\tau}\left(2i\tau r_2[\omega] + r_2'[\omega]\right)}{d[\omega]^2} - 4\frac{d'[\omega]^2}{d[\omega]^4} \tag{9b}$$
$$\Gamma_{21}[\omega] = -4i\tau\frac{d'[\omega]}{d[\omega]^2} + 4\tau^2\frac{1-d[\omega]}{d[\omega]^2} + 2\frac{r_1'[\omega]r_2'[\omega]e^{2i\omega\tau}}{d[\omega]^2} + 4\frac{d'[\omega]^2}{d[\omega]^4} \tag{9c}$$

$\Gamma_{22}$ and $\Gamma_{12} = \Gamma_{21}$ are obtained by exchanging the roles of the two mirrors. At the limit of perfect reflection, the mass corrections (6) are recovered. They are proportional to $\frac{\hbar}{cq}$ in this limit, as it could have been guessed from a dimensional analysis since there are no other dimensioned parameters than $\hbar$, $c$ and $q$. For partially transmitting mirrors, the mass corrections are no longer homogeneous functions of the distance $q$ between the two mirrors, since they now depend upon the reflectivity functions and particularly upon the reflection cutoff frequencies.

## MASS CORRECTION FOR THE COMPOUND SYSTEM

We come to the study of a global motion of the compound system, when the two mirrors move with the same acceleration

$$\delta q_1(t) = \delta q_2(t) = \delta q(t)$$

In other words, their distance remains equal to its initial value

$$q_2(t) - q_1(t) = q_2(0) - q_1(0) = q$$

We notice that the motional force is computed in a first order expansion in the mirrors' displacement, performed in the vicinity of a static configuration (mirrors at rest). In particular, Lorentz contraction, which scales as $\frac{v^2}{c^2}$ with $v$ the velocity of cavity and $c$ the velocity of light, can be disregarded.

The global force exerted upon the cavity is the sum of the forces exerted upon the two mirrors

$$\delta F(t) = \delta F_1(t) + \delta F_2(t)$$

and the motion of the system is described by the linear susceptibility $\chi$ associated with the total force $F$

$$\delta F[\omega] = \chi[\omega]\delta q[\omega] \qquad \chi[\omega] = \sum_i\sum_j \chi_{ij}[\omega]$$

The quasistatic expansion (5) now becomes

$$\delta F(t) = -\left(\kappa \delta q(t) + \lambda \delta q'(t) + \mu \delta q''(t) + \ldots\right)$$
$$\kappa = \sum_i \sum_j \kappa_{ij} \qquad \lambda = \sum_i \sum_j \lambda_{ij} \qquad \mu = \sum_i \sum_j \mu_{ij}$$

The static coefficient $\kappa$ vanishes (see eqs 7), as required by invariance in a global translation of the compound system, or equivalently, by the fact that Casimir force only depends upon the distance between the two mirrors. The viscosity coefficient $\lambda$ also vanishes (see eqs 8), in consistency with Lorentz invariance of vacuum. The force computed for a global motion of the cavity is eventually an inertial force at the quasistatic limit, where the higher order terms are negligible when compared to the second order one

$$\delta F(t) = -\mu \delta q\text{''}(t) \qquad \mu = \sum_i \sum_j \mu_{ij} \tag{10}$$

This relation is exact in the particular case of a uniform acceleration where the higher order terms do vanish.

In the limiting case of perfect mirrors, we obtain from equations (4)

$$\chi[\omega] = \widetilde{\xi}[\omega] + i\xi[\omega] \tag{11a}$$
$$\xi[\omega] = \frac{\hbar}{6\pi c^2}\omega^3 \tag{11b}$$
$$\widetilde{\xi}[\omega] = \frac{\hbar}{6\pi c^2}\left(\omega^3 - \omega\frac{\pi^2}{\tau^2}\right)\tan\frac{\omega\tau}{2} \tag{11c}$$

The dissipative part $\xi$ of the susceptibility is the same for the compound system as for a single perfect mirror. It follows from the fluctuations-dissipation relations that fluctuations of the global force are also the same as for a single mirror. These properties mean that the compound system may, at least for dissipation and fluctuations, be considered as an individual object. They correspond to the fact that the dissipative functions $\xi_{ij}$ coincide precisely in the case of perfect mirrors (see the discussion following eqs 4) with the contributions of outer space.

In contrast, the dispersive part $\widetilde{\xi}$ of the susceptibility differs from the single mirror case. In particular, it contains a mass correction, whereas such a correction was zero for a single mirror

$$\mu = \frac{\chi''[0]}{2} = -\frac{\hbar\pi}{12cq}$$

This means that the field energy stocked inside the cavity, and bound along its motion, contributes to its inertia. Furthermore, the mass correction appears to be directly connected to the Casimir energy $U$ given by equation (3)

$$\mu c^2 = 2U \tag{12}$$

This relation explains the negative sign of the mass correction, since Casimir energy is a binding energy. However, the factor 2 seems to prevent a simple explanation of the mass correction from the law of inertia of energy [3]. A precise discussion requires a more detailed analysis of the law of inertia of energy, and is delayed to the next section.

It is worth recalling that the mass correction $\mu$ has been calculated at the quasistatic limit. The following relation, deduced from equations (11)

$$\widetilde{\xi}[\omega] = \mu\omega^2\left(1 - \left(\frac{\omega\tau}{\pi}\right)^2\right)\frac{\tan\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$$

shows that the motional force behaves as an inertial force at low frequencies only ($\omega\tau \ll 1$ that is $\omega q \ll c$). The dispersive part $\widetilde{\xi}[\omega]$ of the susceptibility presents divergences at the frequencies $\omega = m\frac{\pi}{\tau}$ ($m$ an odd integer except $m = \pm 1$) which result from a constructive interference between the different numbers of cavity roundtrips [8] and may in principle be observable for an arbitrarily small magnitude of the frequency component $\delta q[\omega]$, that is for a velocity of the system remaining much smaller than the velocity of light. In other words, it turns out that the internal optical modes of the Fabry-Perot cavity are coupled to its external mechanical motion, even for a global motion where the distance between the mirrors is constant. At the adiabatic limit, this coupling may be considered as responsible for an inertia correction. At higher frequencies, internal resonances of the Fabry-Perot cavity are efficiently excited and the effective inertia may become large.

For a cavity built with two partially transmitting mirrors, $\mu$ is deduced (see eq. 10) from equations (9)

$$\mu = \frac{i\hbar}{4c^2} \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \omega^2 \left(\Gamma[\omega] - \Gamma[-\omega]\right)$$
$$\Gamma[\omega] = \sum_i \sum_j \Gamma_{ij}[\omega] = -4i\tau \frac{d'[\omega]}{d[\omega]^2}$$

This relation can be transformed by an integration by parts into an expression in terms of the mean Casimir force given by equations (7)

$$\mu = -\frac{2Fq}{c^2} \tag{13a}$$

Expression (12) is recovered at the limit of perfect mirrors, since $(-Fq)$ then coincides with the Casimir energy $U$ ($U$ scales as $\frac{1}{q}$ and $F = \frac{\mathrm{d}U}{\mathrm{d}q}$). In general, the force $F$ is the difference between the mean energy densities inside and outside the cavity, and the quantity $(-Fq)$ is the integral $E_f$ of the field energy density, measured as a difference with respect to the mean density in free space ($e_{\text{inner}}$ and $e_{\text{outer}}$ are the energy densities inside and outside the cavity [2])

$$-Fq = \left(e_{\text{inner}} - e_{\text{outer}}\right) q = E_f \tag{13b}$$

Causal reflectivity functions fulfilling the high frequency transparency requirements have to be frequency dependent. Then, the field experiences reflection delays upon each mirror, and the Casimir energy $U$ may be written as the sum of the integrated field energy $E_f$ and of an extra contribution, attributed to an apparent modification of the cavity length associated with reflection delays (see eq. 27 in ref. [2]). This extra contribution is much smaller than $E_f$ at the limit of short delays, when $\tau$ is greater than the reflexion delays.

For partially transmitting mirrors, the divergences of the motional susceptibilty at the optical resonance frequencies $\omega = m\frac{\pi}{\tau}$ ($m$ an odd integer except $m = \pm 1$) of the cavity are regularized and become dispersion-shaped resonances [8].

## INERTIA OF A STRESSED RIGID BODY

The mass corrections obtained in the foregoing section differ from the expression $\frac{E_f}{c^2}$ which would naively be expected from the law of inertia of energy [3]. It is worth refering this problem to arguments given by Einstein in his first survey article on relativity [14].

The point is that the cavity moves as a rigid body, the two mirrors having the same acceleration, while being submitted to a force. In this situation, the momentum $P$ of the cavity has to be written [19]

$$P = (m + \delta m)\, v \tag{14a}$$

where $v$ is the global velocity $v = q_1' = q_2'$, $m$ the genuine mass of the mirrors, and $\delta m$ a mass correction which depends upon the field energy $E_f$ and the force $F$

$$\delta m = \frac{E_f - Fq}{c^2} \tag{14b}$$

The global force exerted upon the system is therefore

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (m + \delta m)\, a \tag{14c}$$

where $m$ and $\delta m$ are considered as constant and $a = v'$ is the global acceleration. It thus follows from relativistic considerations (and not from the particular dependence of $F$ or $E_f$ versus $q$) that inertia of a stressed rigid body is not given by the simple expression $\frac{E_f}{c^2}$, but by the more elaborate one (14) which involves the value of the stress. This refinement plays a role in the relativistic analysis of a thermodynamic system with a homogeneous normal pressure [20,21].

For the problem studied in the present paper, the stress is the Casimir force and the quantities $E_f$ and $(-Fq)$ coincide, so that the mass given by equations (13) and computed from the motional susceptibility fits expressions (14). This proves not only that Casimir energy does contribute to inertia of the cavity, but that its contribution is precisely what is expected from the law of inertia of energy.

It has however to be emphasized that the usual expression $\frac{E_f}{c^2}$ effectively describes the inertia correction associated with Casimir energy, when the motion of the relativistic center of inertia of the whole system (mirrors and stocked fields) is considered. This follows directly from the first statement of the law of inertia of energy [3], which is of course equivalent to the second statement corresponding to equations (14), as it can be checked by defining the energy $E$, the momentum $P$, and the relativistic center of inertia $Q$ of the compound system

$$E = e_1 + e_2 + E_f \qquad P = p_1 + p_2 + P_f$$
$$EQ = e_1 q_1 + e_2 q_2 + E_f \frac{q_1 + q_2}{2} \tag{15a}$$

where $e_i$ and $p_i$ are the relativistic energy and momentum of the mirror $i = 1, 2$; $E_f$ and $P_f$ are the energy and momentum of the stocked field; we have used the fact that the stocked field energy is distributed homogeneously inside the cavity and has therefore its center of inertia at the middle point $\frac{q_1+q_2}{2}$. Computing explicitly the time derivative of the center of inertia $Q$ defined above and noting that

$$p_i = e_i \frac{q_i'}{c^2} \qquad e_i' = p_i' q_i' = F_i q_i' \qquad E' = 0$$

one shows that equations (14) are equivalent to

$$c^2 P = e_1 q_1' + e_2 q_2' + (E_f - Fq) \frac{q_1' + q_2'}{2} = EQ' \tag{15b}$$

The center of inertia $Q$ thus behaves as the position of a particle of mass $\frac{E}{c^2}$.

## DISCUSSION

The main result obtained in the present paper is that Casimir energy, that is a change in vacuum energy, does contribute to the inertial mass of a Fabry-Perot cavity. The computed mass agrees with the prediction of the law of inertia of energy, when the fact that the cavity moves as a stressed rigid body is accounted for. The equivalence principle then tells us that Casimir energy has also to contribute to gravity.

The calculations have been performed in the simple case of a scalar field in a 2D spacetime, and they would have to be generalized to scatterers and vacuum fields in a 4D spacetime. In their present form however, they already meet interesting questions concerning the nature of inertia.

Einstein suggestively stated [3] that *"radiation conveys inertia between emitting and absorbing bodies"*. In the context of the present paper, it must be understood that vacuum fields stocked inside the cavity convey inertia between the two mirrors. The stocked energy is bound to intracavity space, between the two mirrors, rather than to the mirrors themselves (this appears clearly in eqs 15). This is why its contribution to inertia gives rise to properties usually associated with the "principle of relativity of inertia".

Following ideas expressed by Mach, Einstein [22] attempted to define such a principle while he was developing his theory of general relativity. Later on [23], he described as follows the properties associated with such a conception: (1) *The inertia of a body must increase when ponderable masses are piled up in his neighborhood*; (2) *A body must experience an accelerating force when neighboring masses are accelerated;* a third property, concerning rotation, can be disregarded in comparing with calculations in a 2D spacetime. Such effects are actually predicted by general relativity, with a very small magnitude however [24].

Although they are derived from a theoretical study of vacuum fluctuations rather than from a study of gravity, the results of the present paper partly fit these requirements. Indeed, the mass of a scatterer in vacuum is modified by the presence of another scatterer, and the correction depends upon the relative distance $q$ of the two scatterers. Each scatterer experiences a force proportional to the acceleration of the other one. Note that a global acceleration of the whole system gives rise to a force, in agreement with the law of inertia of energy.

As a consequence, it appears fruitful to consider the vacuum, the quantum "empty space", as a Lorentz-invariant realization of inertial reference frame [25], the inertial forces representing the reaction of vacuum fields to an accelerated motion with respect to them. The results of the present paper make clear that this conception is pertinent for those inertial forces which are associated with a stocked field energy like the Casimir energy. An appealing feature of this conception is that it would make plausible that gravity forces are actually equivalent to a modification of vacuum fields [26]. It has nevertheless to be acknowledged that difficulties plague the possibility of explaining all inertia along these lines.

It appears that the effects discussed in the present paper depend on the scattering properties of neighboring scatterers, while no direct relation has been established between these scattering properties and the masses. Furthermore, the magnitude and sign of the mass corrections do not fit Einstein's requirements. Negative mass corrections are obtained, as expected from the fact that Casimir energy is a binding energy. Then, the mass corrections obtained in the present paper scale as $\frac{\hbar}{cq}$ for a perfect mirror; they are negligible when compared to the mirror's mass $m$, as soon as the distance $q$ is greater than the Compton wavelength $\frac{\hbar}{mc}$. It can nevertheless be noted that, if the magnitude of the effect is the same in a 4D spacetime (this would be consistent with dimensional analysis), the mass corrections may become large and even diverge, when scatterers uniformly distributed in space are considered. The effect of the finite time of flight between bodies has to be kept in mind, since distant scatterers can only modify the inertial force with large time delays.

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[20] In equation (14.b), $(E_f - Fq)$ is replaced by $(E_f + pV)$, where $p$ is the pressure and $V$ the volume (see ref. [14]); the change of sign is due to the fact that a positive pressure $p$ represents a repulsion between the two points of application, while a positive force $F$ is defined here as an attraction.

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